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Transient internal waves produced by a moving body in a tank of density-stratified fluid

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The internal waves produced by a moving body are generally longer in the direction of motion than the corresponding surface waves. This difference is accentuated when the density variation is slight and the body velocity is large in which case a very long towing tank may be required for the simulation of a steady-state condition. The following theoretical study of transient waves is intended as a step in relating test conditions and requisite towing-tank sizes.

A source–sink pair travelling for a finite time is used to represent the restricted motion of a body in a tank. The approximate length and volume of the body are fixed, but its precise shape (somewhat irregular and slightly time dependent) is assumed to be of secondary importance and is not calculated here. The density-stratified fluid is assumed to have a constant Brunt–Väisälä frequency.

A solution in the form of a triple sum over the tank eigenfunctions applies quite generally for the internal wave system (neglecting surface waves and the potential-flow-type solution near the body). Examples covering the large-scale structure of the flow field have been solved for two values of an approximate similarity parameter. The value of the similarity parameter indicates how closely steady-state conditions are approached. The first (larger) value chosen produces a well-defined quasi-steady state near the body with transient fluctuations of the order of $\pm 10\,\%$. The second (smaller) value gives a poorly defined quasi-steady state with fluctuations of the order of $\pm 50\,\%$. More elaborate studies varying the tank length, width and depth could be made by programming the calculations.

The effect of a collapsing wake has not been considered here, but might possibly be treated by similar methods.

1. Introduction and preliminary description of methods†

Significance of the transient problem

In studying theoretically the internal waves created by a moving body in a tank of infinite length it was observed that characteristic wavelengths in the direction of motion may be very much greater than the body length. In such cases the asymptotic (far-field) solutions may apply only at very large distances behind the model. The internal wave system developed might then be altered by the

† See appendix for remarks on the choice of methods.

restricted length of an actual towing tank. To investigate this effect a transient problem must be solved.

The problem studied here is the effect of moving a body at constant velocity for a prescribed distance in a tank of finite dimensions. The body is represented by a source–sink pair on the line of motion. The approximate size and length of the body are thus fixed, but the precise shape (somewhat irregular and slightly time dependent) is assumed to be of secondary importance, and is not calculated here.†

The pulsating moving source

It is necessary to construct a source which moves through the fluid at constant velocity, but has zero strength except for a prescribed time. Such a source can be constructed from pulsating sources whose outflow varies in a simple harmonic fashion with time. A Fourier integral process with integration over all frequencies of pulsating sources will later be used to give the required source of finite lifetime.

Construction of a moving pulsating point source

To construct a point source in the tank we first divide the fluid by passing an imaginary horizontal plane through the source location, and we consider separately periodic solutions of the partial differential equation in the upper and the lower fluid. These solutions are chosen so that pressure is continuous across the plane but vertical velocity is not. Thus continuity is not satisfied and periodic distributions of source strength appear in the plane.

These distributions are adjusted to have the form of standing waves (in the moving co-ordinate system) of fluid-particle velocity normal to the plane. The sources created will then be of conventional‡ type (i.e. giving the illusion of motion by activating in sequence a set of stationary sources along the line of motion; see Garrick 1957). This choice ensures that the source–sink pair ultimately used will produce a body whose shape is almost fixed throughout its travel.

Wavenumbers in both the lateral and longitudinal directions are quantized to discrete sets (since there are both end walls and side walls). By summing over these wavenumbers with a double Fourier series, a doubly infinite array of delta functions is constructed. This represents a moving pulsating point source with a doubly infinite image system.

The complete image system

The infinite set of lateral images immediately provides planes of symmetry on which the tank side walls can be placed. To obtain planes of symmetry in which the end walls can be inserted is a little more difficult, since the body changes its distance from the end walls. For this reason it is necessary to superimpose on the forward-moving system a similar system moving in the opposite direction. Then

† See appendix.

‡ Moving pulsating sources are of many types. In a previous paper (Graham & Graham 1971*b*) there is some discussion of this in connexion with acoustical problems.

halfway between a forward-moving body and a backward-moving body (or image) there will be a plane of symmetry at rest with respect to the fluid in which an end wall can be inserted.

The body moving at uniform velocity for a finite time (complete solution)

Using a Fourier integral process and integrating over all frequencies a source of constant strength, existing for a finite time, is constructed. This source has the image system previously discussed, and so may be regarded as a source which moves for a prescribed time or distance in a tank with end walls as well as side walls. The addition of a displaced sink then gives our approximate representation of a submerged body which starts moving at a particular time, moves through a given distance at uniform velocity, then stops. The shape of this body is not affected by the transient nature of the source and sink. However, the shape does vary somewhat with time because of the changing proximity of the end walls and because of very short wavelength, transient internal waves.

The solution now available is a 'particular integral' for the problem. The most general solution must also include a 'complementary function' made up of the most general combination of eigenfunctions for the tank as a whole. The particular-integral solution can be evaluated at large negative times by asymptotic methods. This yields a unique combination of eigenfunctions. Since the fluid in the tank must be completely undisturbed at large negative times (i.e. before the body is moved), the complementary solution must merely cancel out this unique combination of eigenfunctions. The complete solution is now determined.

Discussion of the complete solution

The complementary function was chosen to cancel the particular integral at large negative times, since the fluid must be completely undisturbed then. However, the fluid is also undisturbed at any time before the body is put in motion, so the asymptotic evaluation of the particular integral must be precisely correct until the body motion starts. Similarly the asymptotic evaluation of the complete solution at large *positive* times must be exact at any time after the body motion stops. This materially simplifies the theoretical results for wave motions in the tank after the body has come to rest.

The reason for this simplification is that immediately after the body stops moving the fluid motion in the tank must be composed entirely of the eigenfunctions of the tank as a whole. Each mode has been excited to a certain amplitude and, in the absence of any additional external force, this amplitude must be maintained indefinitely (within the limitations of our linear inviscid theory).

2. Development of equations

The pulsating source and images

The density-stratified fluid is assumed to have a constant Brunt–Väisälä frequency N, defined by

$$N^2 = -g\bar{\rho}_z/\bar{\rho}. \tag{1}$$

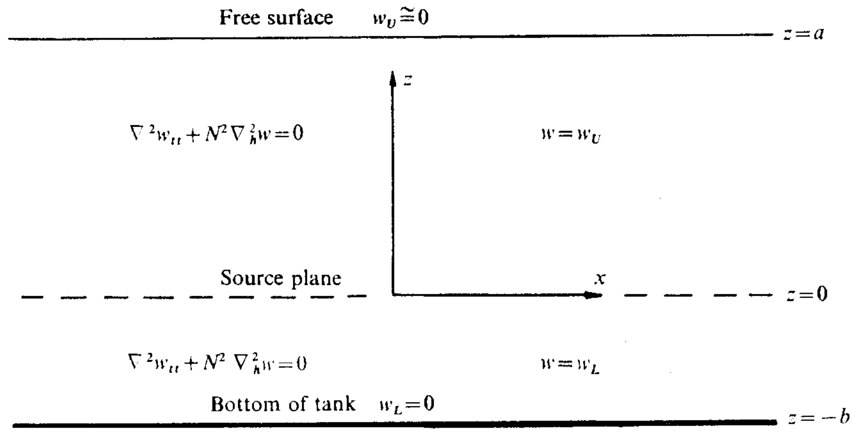


FIGURE 1. Co-ordinate system and geometry.

Here g is the acceleration due to gravity, $\bar{\rho}$ is the steady-state mass density of the fluid and $\bar{\rho}_z$ is the derivative of $\bar{\rho}$ with respect to z (vertical distance). The co-ordinates fixed in the tank are x, y and z in the longitudinal, lateral and vertical directions respectively, and t represents time. At the surface of the fluid $z = a$, at the bottom of the tank $z = -b$ and in the horizontal plane containing the source and images $z = 0$ (see figure 1).

In the regions above and below the source plane the partial differential equation to be satisfied is

$$\nabla^2 w_{tt} + N^2 \nabla_h^2 w = 0, \tag{2}$$

where ∇^2 is the Laplacian operator, ∇_h^2 is the two-dimensional Laplacian operator in the horizontal plane and w is the vertical velocity of the fluid. This equation was presented, for example, by Phillips (1969, p. 162, equation 5.2.7). The secondary effect of density variation on fluid inertia has been neglected compared with its primary effect in producing a restoring force for vertical internal motions. This is the Boussinesq approximation (see Phillips 1969, p. 14, § 2.4). Equation (2) is readily derived by applying small perturbations to the fundamental equations given by Lamb (1945, pp. 4 and 6). See Graham & Graham (1971*a*).

The vertical velocity of the fluid is designated w_U above the source plane and w_L below. For internal waves w_U must be approximately zero at the free surface (see Miles 1971, p. 66). At the bottom of the tank w_L must be zero. Across the source plane the perturbation pressure Δp must be continuous, where Δp is given by

$$\Delta p = -\bar{p} w_{zt}/k^2. \tag{3}$$

It can easily be verified that the following expressions for w_U and w_L satisfy these requirements:

$$w_U = \frac{\nu(\omega)}{2L_1 L_2} \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{\epsilon_n \cos(k\alpha b) \sin[k\alpha(a-z)] \cos(k_1\xi - \omega t) \cos(k_2 y)}{\sin[k\alpha(a+b)]}, \tag{4}$$

$$w_L = -\frac{\nu(\omega)}{2L_1 L_2} \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{\epsilon_n \cos(k\alpha a) \sin[k\alpha(b+z)] \cos(k_1\xi - \omega t) \cos(k_2 y)}{\sin[k\alpha(a+b)]}. \tag{5}$$

Here $k_1 = m\pi/L_1$ and $k_2 = n\pi/L_2$ are wavenumbers in the x and y directions,

$k = (k_1^2+k_2^2)^{\frac{1}{2}}$, $\alpha = [N^2/(\omega+Uk_1)^2-1]^{\frac{1}{2}}$, $\xi = x-Ut$ and $\epsilon_n = \frac{1}{2}$ for $n = 0$ while $\epsilon_n = 1$ for $n > 0$. ξ is a longitudinal co-ordinate measured to the right from the moving pulsating source (of frequency ω), which moves to the right at velocity U. If we now evaluate $w_U - w_L$ at $z = 0$ (i.e. in the source plane) we get

$$w_U - w_L = \frac{\nu(\omega)}{L_1 L_2} \cos(\omega t) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \epsilon_{mn} \cos(m\pi\xi/L_1) \cos(n\pi y/L_2), \tag{6}$$

where $\epsilon_{mn} = \frac{1}{4}$ for $m = n = 0$, $\epsilon_{mn} = \frac{1}{2}$ for $m = 0$ or $n = 0$ with $m \neq n$ and $\epsilon_{mn} = 1$ for $m, n > 0$.

In the m summation, positive and negative terms have been combined to illustrate the superposition of waves moving to the right and waves moving to the left. This superposition produces standing waves in the ξ co-ordinate which sum to represent a doubly infinite array of sources moving to the right with velocity U. (The strength ν is the maximum volume per unit time introduced by any one source.) These sources are spaced $2L_1$ apart in the ξ direction and $2L_2$ apart in the y direction. Lateral symmetry permits the insertion of tank side walls spaced $2L_2$ apart. However, the insertion of end walls must await the introduction of a reverse-motion array of sources.

The sources which have been introduced here are pulsating as well as moving relative to the surrounding fluid. We have chosen the 'conventional' type† of moving source which is constructed from standing waves (in the moving co-ordinate system) of fluid-particle velocity normal to the source plane.

The source which exists for a finite time

In order to construct a source (and image system) of constant strength which exists for a finite time, it is necessary to integrate over all values of ω. If the expressions for w_U and w_L are written in such a general fashion and $w_U - w_L$ is evaluated at $z = 0$, we get

$$w_U - w_L = \frac{1}{L_1 L_2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \epsilon_{mn} \cos(k_1\xi) \cos(k_2 y) \frac{1}{2} \int_{-\infty}^{\infty} \nu^*(\omega) \cos(\omega t)\, d\omega, \tag{7}$$

where $\nu^* d\omega$ now has the dimensions of volume/time. Let $t = 0$ correspond to the centre of travel for the source (not necessarily the centre of the tank, which is later determined by the reverse solution added). If the source strength is to be zero except in the time interval $-\frac{1}{2}\tau < t < \frac{1}{2}\tau$, where the strength is to be ν_0, then $\nu^*(\omega)$ is given by

$$\nu^*(\omega) = \frac{\nu_0}{\pi} \frac{\sin(\frac{1}{2}\omega\tau)}{\frac{1}{2}\omega} \tag{8}$$

(see Sommerfeld 1949, p. 297). The expression for w_U for a source at $\xi = 0$ (and image system) travelling from left to right at velocity U then becomes

$$w_U = \frac{\nu_0}{2\pi L_1 L_2} \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \epsilon_n \cos(k_2 y) \times \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin(\frac{1}{2}\omega\tau) \cos(k\alpha b) \sin[k\alpha(a-z)] \cos(k_1\xi - \omega t)\, d\omega}{\frac{1}{2}\omega \sin[k\alpha(a+b)]}. \tag{9}$$

† See previous discussion.

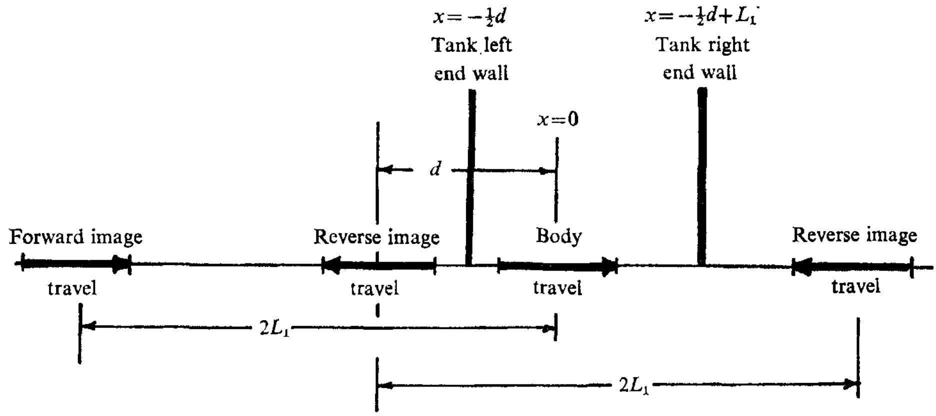


FIGURE 2. Forward and reverse image systems. $\xi = x - Ut$, $\xi = 0$ is body centre, $t = 0$ is mid-time and $x = 0$ is mid-point in travel of body centre.

Introduction of closed body and tank end walls

For a closed body of length approximately l we need a source at $\xi = +\frac{1}{2}l$ and a sink at $\xi = -\frac{1}{2}l$. For this change

$$\cos(k_1\xi - \omega t) \to 2\sin(k_1\xi - \omega t)\sin(\tfrac{1}{2}k_1 l). \tag{10}$$

In order to get planes of symmetry in which the tank end walls can be placed (at $x = -\frac{1}{2}d$ and $x = -\frac{1}{2}d + L_1$) we now need to introduce a source (and image system) moving in the reverse direction with its centre of travel displaced a distance d to the left (see figure 2). We simply ask for a solution giving the same w_U at $x = -a' - d$ as the forward solution gives at $x = a'$. With this change we get

$$\cos(k_1\xi - \omega t) \to -4\sin(\tfrac{1}{2}k_1 l)\cos(k_1 x')\sin[k_1(x_s' + Ut') + \omega(t' - \tfrac{1}{2}\tau)], \tag{11}$$

where x' is distance measured from the left tank end wall, x_s' is the distance of the body starting point from the left tank end wall and t' is time measured from the starting time of the body.

Replacing the $\cos(k_1\xi - \omega t)$ in (9) with the expression in (11) gives

$$w_U = \frac{-2\nu_0}{\pi L_1 L_2}\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}\epsilon_n\cos(k_2 y)\sin(\tfrac{1}{2}k_1 l)\cos(k_1 x')$$
$$\times\frac{1}{2}\int_{-\infty}^{\infty}\frac{\sin(\frac{1}{2}\omega\tau)\cos(k\alpha b)\sin[k\alpha(a-z)]\sin[k_1(x_s' + Ut') + \omega(t' - \frac{1}{2}\tau)]\,d\omega}{\frac{1}{2}\omega\sin[k\alpha(a+b)]}. \tag{12}$$

This applies to a closed body of length approximately l which moves at a uniform velocity U for a prescribed time in a tank of length L_1 and width $2L_2$. The motion starts at a distance x_s' from the left end wall at time $t' = 0$ and continues until $t' = \tau$.

In figure 3 the k_1, ω plane is shown. The combined summation over k_1 (or m) and

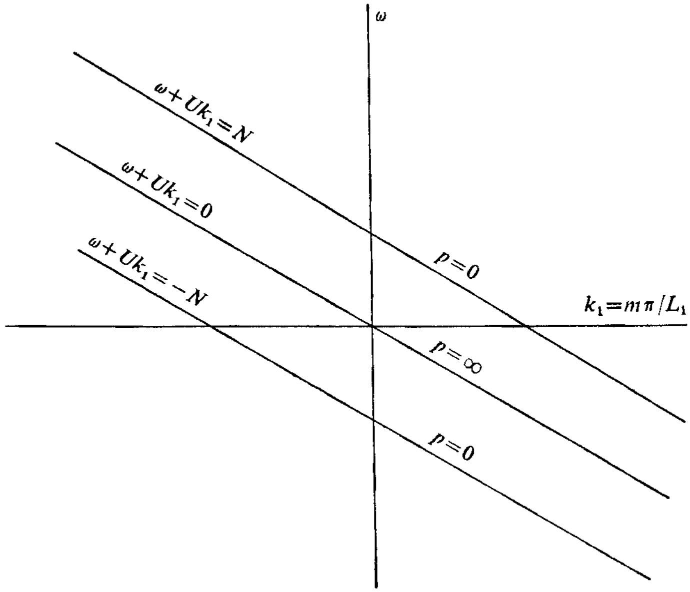


FIGURE 3. The k_1, ω plane.

integration over ω can be carried out more readily in skewed co-ordinates, and we rewrite w_U as

$$w_U = \frac{-2\nu_0}{\pi L_1 L_2} \sum_{n=0}^{\infty} \epsilon_n \cos(k_2 y) \sum_{m=-\infty}^{\infty} \sin(\tfrac{1}{2}k_1 l)\cos(k_1 x')$$
$$\times \left\{ \int_{-Uk_1}^{N-Uk_1} \frac{\cos(k\alpha b)\sin[k\alpha(a-z)]\sin(\tfrac{1}{2}\omega\tau)\sin[k_1(x_s'+Ut')+\omega(t'-\tfrac{1}{2}\tau)]\,d\omega}{\tfrac{1}{2}\omega\sin[k\alpha(a+b)]} \right.$$
$$\left. + \int_{N-Uk_1}^{\infty} \frac{\cosh(k\alpha' b)\sinh[k\alpha'(a-z)]\sin(\tfrac{1}{2}\omega\tau)\sin[k_1(x_s'+Ut')+\omega(t'-\tfrac{1}{2}\tau)]\,d\omega}{\tfrac{1}{2}\omega\sinh[k\alpha'(a+b)]} \right\}, \quad (13)$$

where $$\alpha = [N^2/(\omega+Uk_1)^2-1]^{\frac{1}{2}}, \quad \alpha' = [1-N^2/(\omega+Uk_1)^2]^{\frac{1}{2}} \quad (14)$$

and, using symmetry,† the summation and integration are performed in the half-plane above $\omega+Uk_1 = 0$ and doubled. Equation (13) corresponds to a 'particular integral', and a 'complementary function' must be added to ensure that the fluid is completely undisturbed at large negative times (i.e. before the body was put in motion). To do this we first evaluate (13) as t' approaches $-\infty$.

Evaluation as $t' \to -\infty$

The integrand now consists of a slowly varying function multiplying a function which oscillates rapidly about zero. Such functions interact weakly, and substantial contributions to the integral occur (*a*) when the rapidly varying function ceases to be rapidly varying (e.g. at some stationary-phase points), or (*b*) when

† Corresponding to reflexion in the origin.

the slowly varying function becomes rapidly varying (i.e. at singular points). This case is of the latter type,† so we investigate the singular points defined by

$$\left.\begin{aligned} &\sin[k\alpha(a+b)] = 0, \\ \text{or} \qquad &k(a+b)[N^2/(\omega_p+Uk_1)^2-1]^{\frac{1}{2}} = p\pi \\ \text{or} \qquad &\omega_p+Uk_1 = N/[1+p^2\pi^2/k^2(a+b)^2]^{\frac{1}{2}}. \end{aligned}\right\} \tag{15}$$

If we let $\omega = \omega_p+\delta$ the contribution of any one singularity to the first integral in (13) becomes

$$\frac{\sin(\frac{1}{2}\omega_p\tau)}{\frac{1}{2}\omega_p}\cos\left(\frac{bp\pi}{a+b}\right)\sin\left[\frac{(a-z)p\pi}{(a+b)}\right]\cos\left[k_1(x_s'+\tfrac{1}{2}U\tau)+\frac{N(t'-\frac{1}{2}\tau)}{[1+p^2\pi^2/k^2(a+b)^2]^{\frac{1}{2}}}\right]$$
$$\times\int_{-\epsilon}^{\epsilon}\frac{\sin[\delta(t'-\frac{1}{2}\tau)]\,d\delta}{[\partial\{\sin[k(a+b)(N^2/(\omega+Uk_1)^2-1)^{\frac{1}{2}}]\}/\partial\omega]_{\omega=\omega_p}\delta}. \tag{16}$$

For $t'\to-\infty$ (with τ, the total time of the body motion, finite)

$$\int_{-\epsilon}^{\epsilon}\frac{\sin[\delta(t'-\frac{1}{2}\tau)]\,d\delta}{\delta} = -\pi. \tag{17}$$

Then for large negative times or, in fact, any time prior to $t'=0$, w_U becomes‡

$$\begin{aligned} w_U = {} & \frac{-4\pi\nu_0 N}{L_1L_2(a+b)^2}\sum_{n=0}^{\infty}\epsilon_n\cos(k_2y)\sum_{m=0}^{\infty}\sin(\tfrac{1}{2}k_1l)\cos(k_1x') \\ & \times\sum_{p=0}^{\infty}\frac{(p/k^2)\cos[bp\pi/(a+b)]\sin[(a-z)p\pi/(a+b)]}{\cos(p\pi)}\frac{f^3}{N^3} \\ & \times\left[\frac{1}{2}\frac{\sin[f(\tau-t')-k_1(x_s'+\tau U)]+\sin[k_1x_s'+ft']}{f-Uk_1}\right. \\ & \left.-\frac{1}{2}\frac{\sin[f(\tau-t')+k_1(x_s'+\tau U)]-\sin[k_1x_s'-ft']}{f+Uk_1}\right]. \end{aligned} \tag{18}$$

Here $f = N/[1+p^2\pi^2/k^2(a+b)^2]^{\frac{1}{2}}$ and is the natural frequency of the tank mode designated by k and p (or, by m, n and p). ν_0, the source strength, may be replaced by UA_B, where A_B is the maximum cross-sectional area of the body. Note again that $\epsilon_n=\frac{1}{2}$ for $n=0$, $\epsilon_n=1$ for $n>0$, L_1 = tank length, $2L_2$ = tank width, $a+b$ = tank depth, l = body length, $k_1=m\pi/L_1$, $k_2=n\pi/L_2$, $k^2=(k_1^2+k_2^2)$, the body is in the plane $z=0$, x' = distance of observation point from left end wall, x_s' = distance of starting point from left end wall, t' = time measured from starting time and τ = total time of motion (see figure 4).

For $N/[1+p^2\pi^2/k^2(a+b)^2]^{\frac{1}{2}}-Uk_1\simeq 0$ waves travel at nearly the body velocity, producing locally a pseudo-steady-state flow pattern relative to the body. Actually each natural tank mode considered individually is a standing wave. Travelling waves occur locally when two adjacent modes of similar amplitude approach a 90° phase-displacement in space and time. Only in a tank of infinite extent (relative to the region of interest) are the modes so closely spaced that one can speak of a travelling wave of fixed wavenumber.

† See appendix.
‡ For any time after $t'=\tau$, $w_U = -2\times$ (value given by (18)), as noted later.

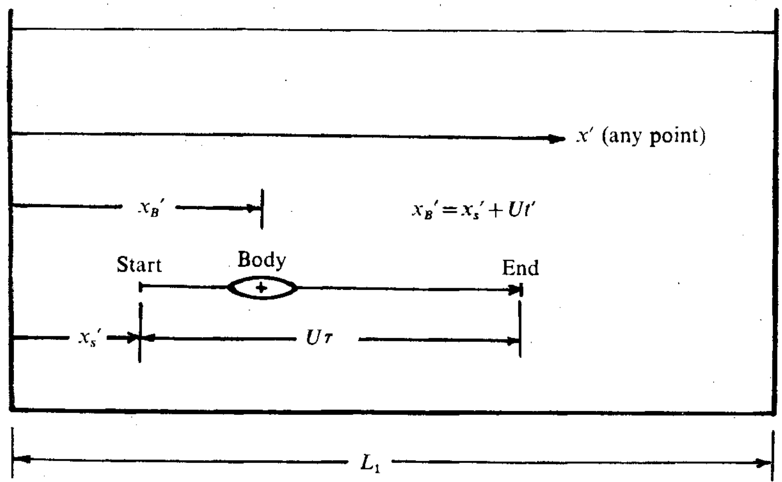


FIGURE 4. Final tank co-ordinate system.

3. Discussion of the solution

The complete solution valid above the body for any time t' before, during or after the body motion is obtained by subtracting (18) from (13). As $t' \to +\infty$, or at any time after the body motion ceases, w_U is given by -2 times the right-hand side of (18). This latter expression appears to be the only one needed in practice.† If we wish to examine the internal wave system at some time t' before the body has stopped moving we can make a new calculation for τ just less than t', knowing that any motion of the body subsequent to the observation is irrelevant. The internal wave system is essentially the same immediately before and immediately after the body stops. The very low frequency character of the internal wave system (N being an upper bound to the frequency) prevents an immediate response to a sudden change such as the impulsive stopping of the body. It must be remembered however that the potential-flow-type field near the body collapses instantaneously as t' passes through the value τ.

If we confine our attention strictly to the internal wave system, then (18) (with the factor -2 applied) should give the proper result for any $t' > 0$ and any place in the tank above the body. (A similar expression is easily obtained for the lower region.)

4. Examples

The solution for w_U (equation (18)) is a sum over the triply infinite set of characteristic modes for internal waves in the tank. Each mode is designated by its values of m, n and p, which are proportional to wavenumbers in the longitudinal, lateral and vertical directions. Since the amplitudes of these modes do

† Except for studying the body shape and attendant potential-flow-type field near the body.

not fall off rapidly with increasing m, n and p, the first problem in working out an example is to limit the number of modes to be considered.†

Modes with wavelengths of the order of the body thickness (or less) in the lateral and vertical directions are immediately suspect as to their magnitudes, because they would tend to create a distorted shape for the body, which has been defined only by its approximate length and volume. Similarly, the magnitudes of modes whose wavelengths in the longitudinal direction are as short as the body length are also questionable. To avoid this difficulty it would be necessary to solve a much more complicated problem, prescribing a smooth rigid body shape maintained fixed throughout the body travel. A continuous (time-varying) distribution of sources and sinks throughout a small region of space would probably be required to achieve this. (Any convergence difficulties in the present solution are presumably associated with the fact that the expansion of a delta function is itself not convergent.)

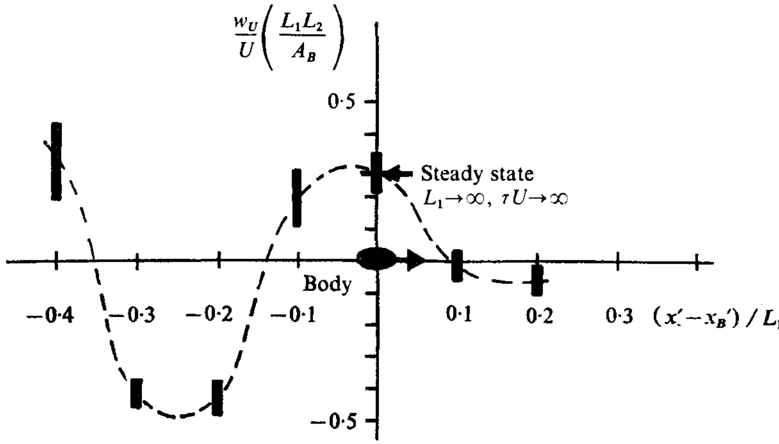
Since the contributions of very short wavelengths must be ignored in this solution, one might consider only the average value of w_U throughout a small region of space. Even more simply, as we have done here, one may arbitrarily discard the short wavelength structure of the flow field by assigning upper limits to m, n and p, and concentrating attention on the longer wavelength structure of the flow field.

The primary problem undertaken here is to determine when a quasi-steady-state condition exists in the tank relative to the body. This quasi-steady state must consist of a wave system which travels at nearly the body speed. No individual tank mode is a travelling wave, but two or more modes of similar wavelength and proper phase relation may simulate a travelling wave locally. Such modes must be approximately 'resonant' with the body speed in the sense that $f \cong Uk_1$. In the following example we first consider modes such that $|f-Uk_1| \leqslant 2\pi U/L_1$. This excludes modes which, oscillating at their natural frequency, would get out of phase with the body motion by more than one complete cycle while the body travels from one end of the tank to the other. (The limit on $|f-Uk_1|$ must be re-examined for other examples.)

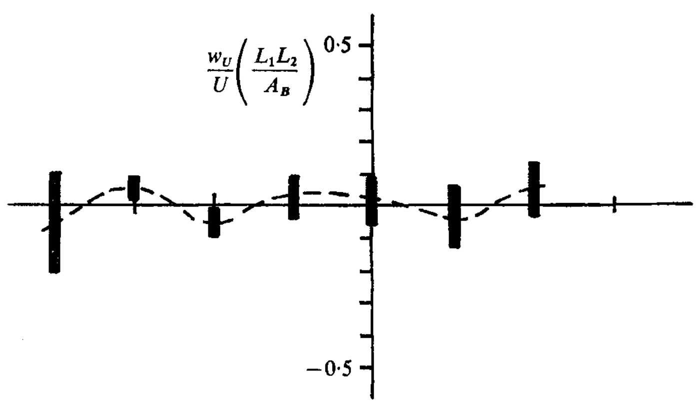
For this example (with L_1 = tank length, L_2 = half-width and $a+b$ = depth) the tank geometry is as follows: $L_1/L_1 = 30$ and $(a+b)/L_2 = 1{\cdot}5$. The Brunt–Väisälä frequency N is defined by $NL_1/\pi U = 5{\cdot}0$. The body length divided by tank length $l/L_1 = \frac{1}{30}$; the body is at the half-depth ($a = b$) and starts at the left end of the tank ($x_s' = 0$). w_U is measured where it is a maximum, halfway between the body level and the surface ($z = \frac{1}{2}a$).

Figure 5(a) shows the effect of considering four modes, $m = 3, 4, 5, 6$, with $p = 2$ and $n = 2$. These are the modes which are approximately resonant with the body speed within the limit previously specified. ($n = 0$ gives a negligible contribution here, and the omission of p and n values greater than 2 indicates that we are examining only the largest wavelength structure of the flow field.) The body starts at the left end of the tank and moves with uniform speed. No observations of w_U are made until the body reaches the middle of the tank, $x_B'/L_1 = 0{\cdot}5$. Then observations are made when the body is at 0·5, 0·6, 0·7 and

† See appendix.



(a)



(b)

FIGURE 5. w_U *vs.* longitudinal distance from body for body at 0·5, 0·6, 0·7 and 0·8 of tank length ($n = 2$, $p = 2$); $z = \frac{1}{2}a$, $y = \frac{1}{2}L_2$. (a) $m = 3, 4, 5, 6$. (b) $m = 7, 8, 9, 10$.

0·8 of the tank length. These observations are made halfway out to the tank side wall ($y = \frac{1}{2}L_2$) and both upstream and downstream of the body. Plotted in coordinates fixed in the body, the values of $w_U L_1 L_2/UA_B$ show a pronounced quasi-steady state relative to the body with secondary transient fluctuations indicated by the length of the vertical bars covering four observations.

In figure 5(b) modes outside the resonant range are studied, $m = 7, 8, 9, 10$, with $p = 2$ and $n = 2$. These give rise to shorter wavelength fluctuations whose dominant characteristic is their transient nature. Second, there is a small steady-state type of contribution.

For comparison in figure 5(a) there is also shown the steady-state value, at $x' = x_B'$, for a body travelling an infinite distance in an infinitely long tank. (The tank width is maintained at $2L_2$ and the depth at $a+b$.) This result is obtainable from (18), by letting L_1 and τU approach infinity, with $t' = \tau$.

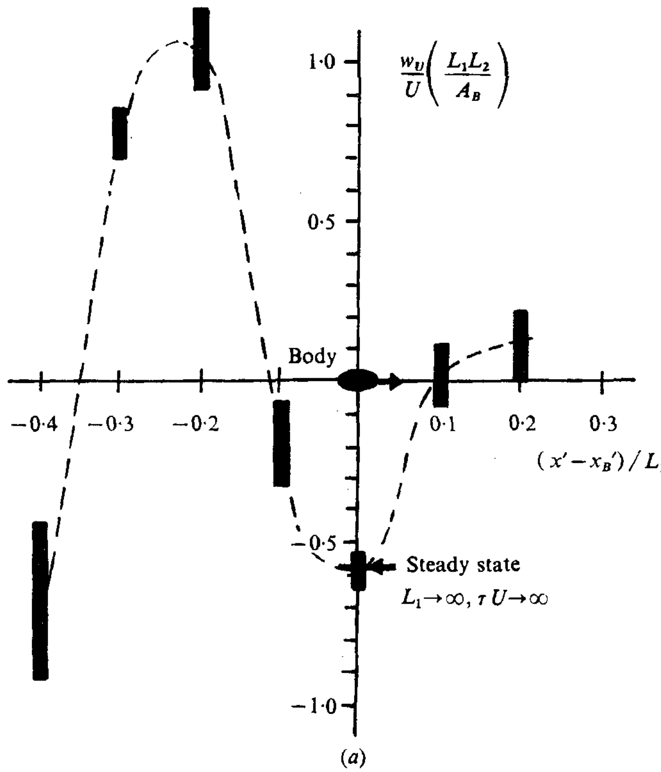


FIGURE 6(a). For legend see facing page.

Alternatively the steady-state result for the infinitely long tank can be obtained from an earlier report (Graham & Graham 1971*a*). In the transverse plane containing the body the expression for w_U is

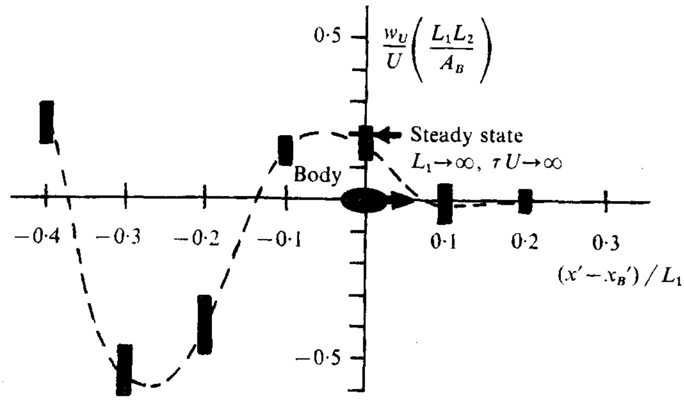
$$\frac{w_U}{U} = \frac{2\pi A_B(-1)^p p\epsilon_n \cos\left(\dfrac{n\pi y}{L_2}\right)\cos\left(\dfrac{bp\pi}{a+b}\right)\sin\left[\dfrac{(a-z)p\pi}{a+b}\right]k_{1c}^3 \sin\left(\tfrac{1}{2}k_{1c}l\right)}{L_2(a+b)^2[k_{1c}^4+(N^2/U^2)(n^2\pi^2/L_2^2)]}.$$

This is for any one mode defined by n and p, and k_{1c} is given by

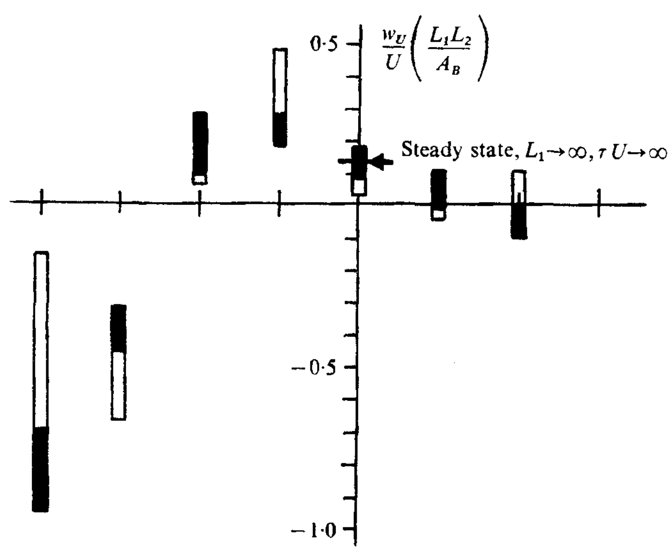
$$k_{1c} = \left\{\frac{1}{2}\left(\left[\frac{n^2\pi^2}{L_2^2}+\frac{p^2\pi^2}{(a+b)^2}-\frac{N^2}{U^2}\right]^2+\frac{4n^2\pi^2}{L_2^2}\frac{N^2}{U^2}\right)^{\frac{1}{2}}-\frac{1}{2}\left[\frac{n^2\pi^2}{L_2^2}+\frac{p^2\pi^2}{(a+b)^2}-\frac{N^2}{U^2}\right]\right\}^{\frac{1}{2}}.$$

The quasi-steady-state results shown in figure 5(*a*) seem to be in good agreement with the steady-state point for the infinitely long tank.

Figures 6(*a*)–(*c*) show the effect of introducing more modes (primarily additional values of n and p) and so including a somewhat smaller wavelength structure of the flow field than in figure 5. However these modes still conform approximately to the requirement that $|f-Uk_1| \leqslant 2\pi U/L_1$ and so are nearly resonant with the body speed. Also, figures 6(*a*)–(*c*) indicate the nature of the flow field at three lateral positions in the tank: $y = 0, \frac{1}{2}L_2$ and L_2. The vertical position of observation is still halfway between the body and the surface, at $z = \frac{1}{2}a$.



(*b*)



(*c*)

FIGURE 6. w_U *vs.* longitudinal distance from body for body at 0·5, 0·6, 0·7 and 0·8 of tank length (n = 0, 1, 2, 3, 4; p = 2, 6); $z = \frac{1}{2}a$, m = 1, 2, 3, 4, 5, 6. (*a*) $y = 0$. (*b*) $y = \frac{1}{2}L_2$. (*c*) $y = L_2$.

$w_U L_1 L_2/UA_B$ is again plotted for positions ahead of and behind the body in body co-ordinates, and the length of the vertical bars indicates the range of values for observations made when the body is at 0·5, 0·6, 0·7 and 0·8 of the tank length.

At $y = 0$ and $y = \frac{1}{2}L_2$ a well-defined quasi-steady state appears. The trough at the body for $y = 0$ is apparently well behind the body for $y = \frac{1}{2}L_2$, so that the inclination of the wave to the line of motion is roughly 4°. At $y = L_2$ the quasi-steady state is not so well defined. Perhaps this should be anticipated because the basic wave system created by the body at the centre-line tends to encounter the side wall far behind the body and near the end of the tank. If the earliest observation (when the body is at 0·5 of the tank length) is omitted the definition seems much better. This is indicated by shading the bars only over the extent of the 0·6, 0·7 and 0·8 observations.

5. The similarity parameter

The parameter $NL_1/\pi U$ serves as an approximate similarity parameter for the large-scale structure of the flow field in many cases of interest. It is necessary that the wave crests be inclined at a small angle to the direction of motion, so that $k \simeq n\pi/L_2$, and that the body length be small relative to the wavelength in the direction of motion. Generally these conditions are satisfied as in the present examples but not, of course, in extreme cases where transversal waves appear.

When the value of $NL_1/\pi U$ is maintained (for example, by halving N and doubling L_1) each tank mode has its amplitude multiplied by the same factor and its wavelength adjusted in proportion to the tank length. The complete wave systems are then similar and the magnitude of the transient oscillations relative to the quasi-steady state is maintained. Thus if N is halved (or U doubled) L_1 must be doubled to maintain the same relation between transient and steady-state values for a given body moving over the same fraction of the tank length. (The tank depth and width and body depth are of course held constant. The above analysis applies when the body is in motion (i.e. when $t' = \tau$). For $t' > \tau$ similarity is preserved by holding $t'N$, τN and $NL_1/\pi U$ constant.)

The effect of reducing the magnitude of $NL_1/\pi U$ is to increase the departure from steady-state conditions. This is illustrated in figure 7, where the average transient amplitude over the maximum quasi-steady-state value is used as a measure of fluctuation. For $NL_1/\pi U = 5$ (corresponding to the preceding examples) this fluctuation is only $\pm 10\%$. For $NL_1/\pi U = 1$ (and other parameters the same as in preceding examples) the fluctuation has grown to $\pm 50\%$. In this latter case $m = 1$ (wavelength $= 2 \times$ tank length) should be the dominant mode but figure 7 (*b*) shows little evidence of this. Thus the length of the tank is clearly inadequate for obtaining a close approach to steady-state conditions in this case. (The limited number of modes considered here still covers the resonant range adequately.)

Appendix. Notes on the methods of analysis

(i) The analytical procedures used here are adopted to facilitate the development from the author's own point of view, which is essentially physical rather than rigorously mathematical. However, the methods, though unconventional, have a sound basis in classical mathematics.

Laplace transforms probably could be used, although the necessity of terminating the body motion as well as initiating it might slightly complicate their use. (Termination of the body motion simplifies the present analysis, and makes possible the investigation of the wave system after the body stops.)

(ii) Convergence questions arise for the triple summation used. However, examination of the *large-scale* structure of the flow field seems quite feasible,

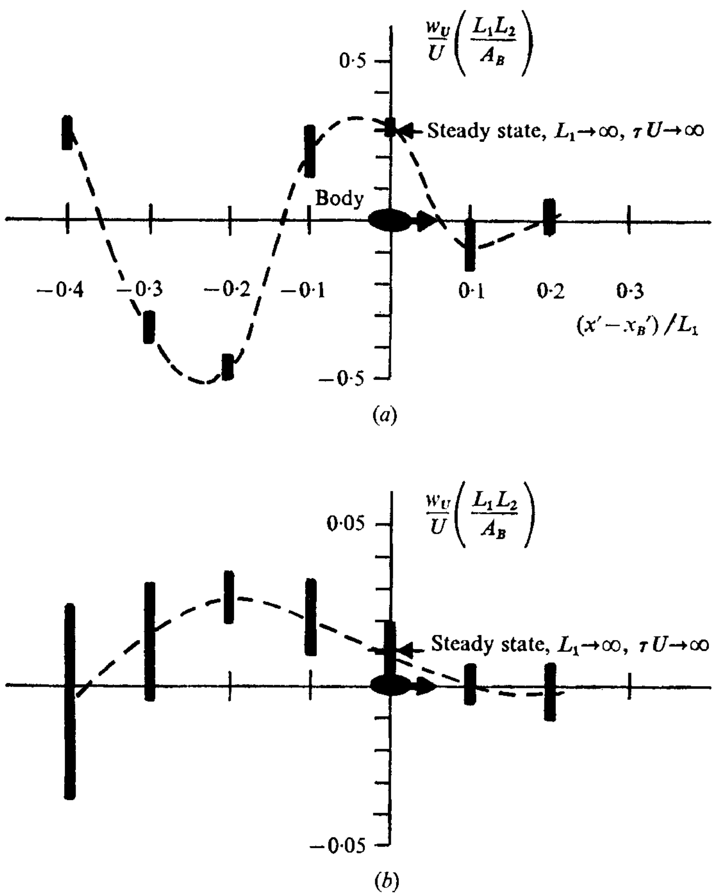


FIGURE 7. Effect of similarity parameter $NL_1/\pi U$ on the transient fluctuations. $n = 2$, $p = 2; y = \frac{1}{2}L_2$. (*a*) $NL_1/\pi U = 5{\cdot}0, m = 3, \ldots, 10$, fluctuation $= \pm 10\,\%$. (*b*) $NL_1/\pi U = 1{\cdot}0$, $m = 1, 2, 3$, fluctuation $= \pm 50\,\%$.

the convergence problem being associated with the many omitted modes of successively smaller wavelengths (i.e. the *fine structure* of the flow field).

(iii) It was pointed out by a referee that, for $|t'|$ large, stationary-phase points may appear in the numerator of the first integral in (13). These points appear near the limits of integration, one of them, for example, corresponding to $\omega \to -Uk_1$, so that $\alpha \to \infty$. However, such analysis leaves a rapidly oscillating function in the denominator, and the criterion for significant contributions to the integral (i.e. that the rapidly varying part of the integrand becomes slowly varying) is not satisfied. More generally, one can perhaps reason as follows. The first integral in (13) can be rewritten as

$$\lim_{\epsilon\to 0}\int_{-Uk_1+\epsilon}^{N-Uk_1-\epsilon} \ldots d\omega.$$

For each ϵ a lower bound can be set on $|t'|$ such that $|t'|$ is always much greater than terms such as $d(k\alpha b)/d\omega$. It is then possible for ϵ to approach zero and $|t'|$ to approach infinity in such a fashion that stationary-phase points never appear within (or on the boundaries of) the region of integration. This applies in the limit $\epsilon \to 0$ since, although $|t'|$ and $d(k\alpha b)/d\omega$ both become infinite, $|t'|$ is, nevertheless, much greater throughout the closed region of integration.

(iv) If the fluid is assumed to be homogeneous, incompressible and of infinite extent, the basic body shape created by a conventional source–sink pair is a Rankine ovoid. Instantaneous creation of the source–sink pair apparently creates this shape instantaneously, according to a preliminary mathematical analysis. This is consistent with the infinite communication speed (speed of sound) in an incompressible fluid.

The body actually created, being 'limp', is slightly distorted by the presence of the tank walls and the large-scale wave structure. This distortion is somewhat time dependent, since transient internal waves exist and the end walls of the tank are not at a fixed distance from the body. The small-scale internal waves, since they create a 'rough' and thoroughly unrealistic body, are discarded. A much more detailed and difficult analysis would probably be required to define the fine structure of the flow field for a smooth rigid body.

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